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On the semimartingale property via bounded logarithmic utility

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Abstract This paper provides a new version of the condition of Di Nunno et al. (2003), Ankirchner and Imkeller (2005) and Biagini and Øksendal (2005) ensuring the semimartingale property for a large class of continuous stochastic processes. Unlike our predecessors, we base our modeling framework on the concept of portfolio proportions which yields a short self-contained proof of the main theorem, as well as a counterexample, showing that analogues of our results do not hold in the discontinuous setting.

Keywords Arbitrage · Enlargement of filtrations · Financial markets · Logarithmic utility · Semimartingales · Stochastic processes · Utility maximization

JEL Classification Numbers C61 · G11

1 Introduction and summary

In [5] the connection between the concept of no-arbitrage and the assumption that the financial assets are driven by semimartingales is initiated. Here it is shown there that if the financial market satisfies the condition of *no free lunch with vanishing risk* for simple trading strategies, then the traded securities

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allow for a semimartingale decomposition. This and similar results depend heavily on the mathematical constructs - the theory of stochastic integration employed, and the class of the integrands used - to describe the economic concept of no-arbitrage. [2] illustrates possible pitfalls resulting from an attempt of economic interpretation of mathematical results based on an integration theory at odds with the financial intuition.

The main result of the present paper is inspired by [3] and [6], and states (loosely) that a continuous process with finite quadratic variation is a semimartingale if the expected utility of a logarithmic investor is uniformly bounded from above over a specific natural class of trading strategies. Unlike [3] and [6], we do not replace the Itô integration with the anticipative *forward* integration, and we do not assume the existence of a trading strategy that achieves the optimal expected logarithmic utility. In fact, the existence of such a strategy is one of the conclusions of our main theorem.

The recent and independent paper [1] develops an idea similar to ours and relates the semimartingality of the stock-price process to the boundedness of the expected utility. The authors base their approach on simple buy-and-hold strategies thereby circumventing the lack of a stochastic integration theory. Our approach is different - it hinges on the observation that a canonical integration theory can be based on simple portfolio *proportions*, without falling into the traps described in [2]. Indeed, while one of the main results of [1] is that bounded utility implies semimartingality, regardless of the continuity properties of the process under scrutiny, our epilogue is different. We construct an example of a discontinuous non-semimartingale S with the property that the expected logarithmic utility is uniformly bounded over all strategies in which portfolio proportions are simple processes. Moreover, there exists a shrinkage of the original filtration under which the process S is a semimartingale. The existence of such a counterexample poses the following question: *Can we describe (and work with) a class of stochastic processes, strictly larger than the class of semimartingales, for which the logarithmic investors will not be able to achieve arbitrarily large expected utilities?* While the non-semimartingales in this class will surely admit free lunch with vanishing risk, the possibiliy of their use in financial modeling is not ruled out. Indeed, the logarithmic investors will not demand unlimited quantities of such securities. We leave this question for further research.

In the continuous case, the flavor of our results agrees with [1], but our approach provides new insights in several respects. First, the proof of our main theorem is short and self-contained, and uses a simple Hilbert-space argument. As a consequence of this, we are able to explicitly derive the semimartingale decomposition of the stock price in terms of the Riesz representation of a suitably defined linear functional. The proof of the related result in [1] is based on the already mentioned result of [5], and provides only the abstract existence of the semimartingale decomposition. Second, a byproduct of our analysis is the existence of the optimal trading strategy for an investor with logarithmic utility - the growth-optimal portfolio.

The paper is structured as follows: In the first section we describe the framework and prove our main result. The second section provides a counterexample which illustrates the fact that, when jumps are present, bounded

logarithmic utility on simple portfolio proportions is not sufficient to grant the semimartingality of the price process.

As all our stochastic processes are defined on the time horizon $[0,1]$, we will consistently use the shorthand S for the process $(S_t)_{t\in[0,1]}$, throughout the paper.

2 The Main Result for Continuous Processes

2.1 The Modeling Framework

We consider a continuous stochastic process S, defined on the unit time horizon $[0,1]$, and adapted to a complete and right-continuous filtration $\mathbb{F} \triangleq (\mathcal{F}_t)_{t\in[0,1]}$, on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We assume that S has finite quadratic variation on $[0,1]$, meaning for $\omega \in \Omega$ the following limit exists

$$\lim_{n\to\infty} \sum_{k=1}^{n} \left(S_{\frac{k}{n}} - S_{\frac{k-1}{n}} \right)^2 \tag{1}$$

and is finite. In that case, the process S defined by

$$[S]_t \triangleq \lim_{n\to\infty} \sum_{k=1}^{n} \left(S_{t\wedge \frac{k}{n}} - S_{t\wedge \frac{k-1}{n}} \right)^2 \tag{2}$$

is finite valued, non-decreasing and continuous. Of course, the sequence $\{0, \frac{1}{n}, \ldots, \frac{n-1}{n}, 1\}$ of partitions in (2) is chosen for simplicity; any other sequence with comparable properties would lead to the same conclusions.

Remark 1 Arguably the most natural way to ensure the existence of the quadratic variation, as defined in (1), is to assume the existence of a filtration $\mathbb{F}' \triangleq (\mathcal{F}'_t)_{t\in[0,1]}$ smaller than $\mathbb{F}$ (i.e., $\mathcal{F}'_t \subseteq \mathcal{F}_t$, $t \in [0,1]$) such that S is an $\mathbb{F}'$-semimartingale. Indeed, semimartingales have finite quadratic variation, which, being defined in a pathwise manner, remains undisturbed under enlargements of the filtration. Inside-trading models typically assume the existence of two classes of investors - *regular investors*, with access to the public information $\mathbb{F}'$, and *insiders*, whose information set is modeled by $\mathbb{F}$, see the seminal paper [7]. In this setting, S represents the stock-price (or the return) process described by a semimartingale from the regular investor's point. Our main result, below, can be applied in this setting as a sufficient condition on the insider's (superior) information structure $\mathbb{F}$, so that S remains a semimartingale under $\mathbb{F}$ as well.

2.2 Some Classes of Stochastic Processes

Let $\mathcal{H}^s$ denote the set of all stochastic processes π of the form:

$$\pi_t = \sum_{i=1}^{n} K_i \mathbf{1}_{(T_{i-1},T_i]}(t), \tag{3}$$

where $n \in \mathbb{N}$, $0 \triangleq T_0 \leq T_1 \leq \cdots \leq T_n \triangleq 1$ are $\mathbb{F}$-stopping times and $K_i \in \mathbb{L}^\infty(\mathcal{F}_{T_{i-1}})$, $i = 1, \ldots, n$.

Remark 2 An analogous class of processes with the filtration $\mathbb{F}$ replaced by a different filtration $\mathbb{G}$ will be needed in Section 3. Such a class will be denoted by $\mathcal{H}^s(\mathbb{G})$.

It is well known that

$$\mathcal{H}^2 \triangleq \{\pi : \pi \text{ is predictable and } ||\pi||_{\mathcal{H}^2} < \infty\},$$

is a Hilbert space where $||\pi||^2_{\mathcal{H}^2} \triangleq \mathbb{E}\int_0^1 \pi_u^2 \, d\,[S]_u$. As no integrability assumptions will be placed on either S or $[S]$, the stopping-time sequence $\{T_n\}_{n=1}^\infty$, where

$$T_n \triangleq \inf\{t \leq 1 : |S_t| > n\} \wedge \inf\{t \leq 1 : [S]_t > n\} \wedge 1, \tag{4}$$

will prove useful in the reduction arguments in the sequel. Indeed, $T_n \leq T_{n+1} \leq 1$ for all n, and $\mathbb{P}(T_n = 1) \to 1$ for $n \to \infty$. Finally, we define $\mathcal{H}^b \triangleq \cup_{n \in \mathbb{N}} \mathcal{H}^b_n$, where

$$\mathcal{H}^b_n \triangleq \{\pi \in \mathcal{H}^2 \,:\, \pi_t = 0, \text{ for } t > T_n\},\ n \in \mathbb{N}.$$

We are now ready to state and prove the following auxiliary result.

Proposition 1 *$\mathcal{H}^b \cap \mathcal{H}^s$ is dense in $\mathcal{H}^2$ with respect to the norm $||\cdot||_{\mathcal{H}^2}$.*

Proof Pick a $\pi \in \mathcal{H}^2$, and note that

$$||\pi - \pi \mathbf{1}_{[0,T_n]}||^2_{\mathcal{H}^2} = \mathbb{E}\left[\int_{T_n}^1 \pi_u^2 \, d\,[S]_u\right] \to 0, \text{ as } n \to \infty$$

by the Dominated Convergence Theorem. Hence, the family $\mathcal{H}^b$ is dense in $\mathcal{H}^2$. To finish the proof of the proposition, it is enough to show that $\mathcal{H}^b \cap \mathcal{H}^s$ is dense in $\mathcal{H}^b$. For this, in turn, it suffices to prove that $\mathcal{H}^b_n \cap \mathcal{H}^s$ is dense in $\mathcal{H}^b_n$, for each n. To verify this statement, we define the process $A_t \triangleq [S]_t^{T_n}$ and apply Lemma 2.7, p. 135 in [8]. $\Diamond$

2.3 The Canonical Definition of Stochastic Exponentials

Although no Itô-type integration theory exists for general adapted integrands with respect to a process S merely satisfying the assumptions of Subsection 2.1, we can always define the stochastic integral for an integrand $\pi \in \mathcal{H}^s$, of the form (3), in the familiar way

$$(\pi \cdot S)_t \triangleq \sum_{i=1}^n K_i \left(S_{T_i \wedge t} - S_{T_{i-1} \wedge t}\right). \tag{5}$$

More importantly for our results, stochastic exponentials can be defined canonically as well by

$$\mathcal{E}(\pi \cdot S)_t \triangleq \exp\left(\int_0^t \pi_u \, dS_u - \frac{1}{2}\int_0^t \pi_u^2 \, d\,[S]_u\right) \tag{6}$$

for all $\pi \in \mathcal{H}^s$ with the dS-integral inside the exponential function defined by (5). For $\pi \in \mathcal{H}^s$, we can show that $\mathcal{E}(\pi \cdot S)$ is the unique pathwise solution Z to the Doléans-Dade stochastic differential equation

$$dZ_t = Z_t \pi_t dS_t, \quad Z_0 = 1. \tag{7}$$

Of course, the integrand $Z\pi$ appearing in (7) is not necessarily in $\mathcal{H}^s$. Nevertheless, the integral $\int_0^t Z_u \pi_u \, dS_u$ exists a.s. as a limit of Riemann sums and equals $Z_t - Z_0$. In order to see this, for a given $\pi \in \mathcal{H}^s$ and a fixed $\omega \in \Omega$, we define the continuous function x on $[0,1]$ by

$$x(t) \triangleq \int_0^t \pi_u dS_u - \frac{1}{2} \int_0^t \pi_u^2 d\,[S]_u \,.$$

The quadratic variation of this deterministic function is given by $[x]\,(t) = \int_0^t \pi_u^2 d\,[S]_u$. The statement now follows from combining these expressions with Exercise 3.13 p. 153 in [10] applied to the function $F(x) \triangleq \exp(x)$.

2.4 A Financial Interpretation

We consider a simple financial market consisting of two assets: one risk-free asset with a zero interest rate, and one risky asset whose price at time t will be denoted by P_t and is given by $P_t \triangleq \mathcal{E}(S)_t$. For any simple process $\pi \in \mathcal{H}^s$ of the form (3), the following equation will be used as a definition of the wealth process of a financial agent investing in the market

$$W_0^\pi \triangleq 1, \quad W_t^\pi \triangleq \mathcal{E}(\pi \cdot S)_t \text{ for } t > 0. \tag{8}$$

We will interpret the value π_t as the proportion of his/her current wealth, the investor has invested in the risky asset at time t. In order to motivate this terminology, let us assume for a second that the return process S is a semimartingale, and hence, $P_t \triangleq \mathcal{E}(S)_t$ in the classical sense. With the process H denoting the number of shares of the risky asset in the investor's self financing portfolio, the investor's wealth evolves according to the following equation

$$dW_t = H_t \, dP_t = H_t P_t \, dS_t = \left(\frac{H_t P_t}{W_t} \right) W_t \, dS_t.$$

This translates exactly into $W_t = \mathcal{E}(\pi \cdot S)_t$, for π given by

$$\pi_t \triangleq \frac{H_t P_t}{W_t}.$$

Of course, the process S in our framework is not assumed to be a semimartingale, so the above discussion cannot be transferred to our setting directly. However, the discussion in Subsection 2.3 implies that such a transformation is indeed feasible, as long as we use only simple processes $\pi \in \mathcal{H}^s$.

Remark 3 A somewhat different interpretation of the equation (8) stems from the alternative assumption that S itself (and not $P = \mathcal{E}(S)$) is the price process of the risky asset. In that case, the equation (8) still describes the evolution of the investor's wealth, but now under that understanding that π_t denotes the number of shares of the risky asset held *per unit of wealth* - a concept less common than that of portfolio proportions described above.

Finally, we impose the following assumption on the risk-aversion characteristics of our financial agent: his/her goal is to invest in such a way as to maximize the expected logarithmic utility of the terminal wealth. The agents with this objective are commonly called log-investors.

2.5 The Main Result

In financial terms, the premise of our main result is that the price process of the risky asset is such that the expected utility of the log-investor is uniformly bounded over all simple portfolio-proportion processes $\pi \in \mathcal{H}^s$, i.e.,

$$\sup_{\pi \in \mathcal{H}^s} \mathbb{E}\left[\log(W_1^\pi)\right] < \infty. \tag{9}$$

In this expression, we implicitly use the convention that if $\pi \in \mathcal{H}^s$ renders both the positive and the negative part of $\log(W_1^\pi)$ non-integrable, we define $\mathbb{E}\left[\log(W_1^\pi)\right] \triangleq -\infty$. This convention is widely used in the theory of utility maximization, see e.g., [1] p. 482.

Our main result in the continuous setting is the following.

Theorem 1 *Let S be a continuous adapted stochastic process with finite quadratic variation in the sense of* (1)*, satisfying the condition* (9)*. Then S is a semimartingale with decomposition*

$$S_t = \hat{S}_t + \int_0^t \alpha_u \, d\left[S\right]_u, \tag{10}$$

where $\hat{S}$ is a local martingale and α is a predictable process in $\mathcal{H}^2$.

Proof For $\pi \in \mathcal{H}^s \cap \mathcal{H}^b$, both integrals $\int_0^1 \pi_u \, dS_u$ and $\int_0^1 \pi_u^2 \, d\left[S\right]_u$ are well-defined and have finite expectations; in particular the linear functional Λ, defined by

$$\Lambda(\pi) \triangleq \mathbb{E}\left[\int_0^1 \pi_u \, dS_u\right], \ \pi \in \mathcal{H}^s \cap \mathcal{H}^b$$

is well-defined and finite valued on $\mathcal{H}^s \cap \mathcal{H}^b$. Assumption (9) grants the existence of a finite constant C such that

$$\mathbb{E}\left[\int_0^1 \pi_u \, dS_u - \frac{1}{2}\int_0^1 \pi_u^2 \, d\left[S\right]_u\right] = \mathbb{E}\left[\log(W_1^\pi)\right] \le C, \text{ for all } \pi \in \mathcal{H}^s \cap \mathcal{H}^b \ .$$

Therefore, Λ admits the following bound

$$\Lambda(\pi) \le C + \frac{1}{2}||\pi||^2_{\mathcal{H}^2},$$

which can be strengthened by noting that

$$\Lambda(\pi) = \frac{1}{\gamma}\Lambda(\gamma\pi) \leq \frac{C}{\gamma} + \frac{\gamma}{2}||\pi||^2_{\mathcal{H}^2}, \text{ for each } \gamma > 0. \tag{11}$$

Minimization the right-most part of (11) with respect to γ yields that

$$|\Lambda(\pi)| \leq \sqrt{2C}||\pi||_{\mathcal{H}^2}, \text{ for all } \pi \in \mathcal{H}^s \cap \mathcal{H}^b.$$

From this we conclude that Λ is a continuous linear functional on $\mathcal{H}^s \cap \mathcal{H}^b$. Proposition 1 states that $\mathcal{H}^s \cap \mathcal{H}^b$ is dense in $\mathcal{H}^2$ with respect to the topology induced by the norm $|| \cdot ||_{\mathcal{H}^2}$. Consequently, the linear functional Λ admits a unique linear and continuous extension to $\mathcal{H}^2$. Riesz's representation theorem guarantees the existence of a process $\alpha \in \mathcal{H}^2$ such that

$$\mathbb{E}\left[\int_0^1 \pi_u \, dS_u\right] = \Lambda(\pi) = \mathbb{E}\left[\int_0^1 \pi_u \alpha_u \, d\left[S\right]_u\right], \text{ for } \pi \in \mathcal{H}^s \cap \mathcal{H}^b. \tag{12}$$

The proof is concluded by showing that the sequence $\{T_n\}_{n\in\mathbb{N}}$ defined by (4) can be used to reduce the continuous adapted process

$$\hat{S}_t \triangleq S_t - \int_0^t \alpha_u d\left[S\right]_u$$

to a martingale. To this end, we let τ be an arbitrary stopping time and define the simple process π^n in $\mathcal{H}^s \cap \mathcal{H}^b$ by $\pi^n_u \triangleq \mathbf{1}_{[0,\tau\wedge T_n]}(u)$. Applying the equality (12) to π^n yields

$$\mathbb{E}\left[S_{\tau\wedge T_n}\right] - S_0 = \mathbb{E}\left[\int_0^{\tau\wedge T_n} \alpha_u \, d\left[S\right]_u\right], \text{ that is, } \mathbb{E}\left[\hat{S}^{T_n}_\tau\right] = S_0. \tag{13}$$

Since (13) holds for all stopping times τ, it follows that $\hat{S}^{T_n}$ is a martingale. Hence, $\{\hat{S}_t\}_{t\in[0,1]}$ is a local martingale and therefore, the process $\{S_t\}_{t\in[0,1]}$ is a semimartingale with the decomposition $S_t = \hat{S}_t + \int_0^t \alpha_u \, d\left[S\right]_u$. $\Diamond$

Corollary 1 (The Growth-Optimal Portfolio) *Under the assumptions of Theorem 1, the wealth process W^π is well-defined in the Itô sense for all $\pi \in \mathcal{H}^2$ and the stronger version of* (9)

$$\sup_{\pi\in\mathcal{H}^2} \mathbb{E}\left[\log(W^\pi_1)\right] < \infty$$

holds. Moreover, the supremum is attained by the process $\alpha \in \mathcal{H}^2$ from (10).

Proof Since $\alpha \in \mathcal{H}^2$, the process $\int_0^t \alpha_u \, d\hat{S}_u$ is a true martingale and hence $\mathbb{E}\left[\log(W^\alpha_1)\right] = \frac{1}{2}\mathbb{E}\int_0^1 \alpha^2_u d\left[S\right]_u$. The equation (12) implies that

$$\begin{aligned}\mathbb{E}\left[\log(W^\pi_1)\right] &= \mathbb{E}\left[\int_0^1 \pi_u \, dS_u\right] - \frac{1}{2}\,\mathbb{E}\left[\int_0^1 \pi^2_u \, d\left[S\right]_u\right] \\ &= \mathbb{E}\left[\int_0^1 \left(\pi_u\alpha_u - \frac{1}{2}\pi^2_u\right) d\left[S\right]_u\right] \leq \frac{1}{2}\,\mathbb{E}\left[\int_0^1 \alpha^2_u \, d\left[S\right]_u\right] = \mathbb{E}\left[\log(W^\alpha_1)\right],\end{aligned}$$

for any $\pi \in \mathcal{H}^s \cap \mathcal{H}^b$. It suffices now to use the density of $\mathcal{H}^s \cap \mathcal{H}^b$ in $\mathcal{H}^2$. $\Diamond$

Remark 4 A simple sufficient condition insuring that the local martingale $\hat{S}$ in the semimartingale decomposition (10) is a true martingale is that $\mathbb{E}\left[[S]_1\right] < \infty$. In that case, $\hat{S}$ will be a square-integrable martingale as well.

3 Processes With Jumps

3.1 Definition of the Wealth Process

In this section we investigate whether it is possible to extend the results of Theorem 1 to the case when the stochastic process S admits jumps. We are facing the same problem as in the previous section, i.e., the non-existence of the canonical theory of stochastic integration for non-semimartingales. However, with the motivation from Subsection 2.3, a canonical definition of the stochastic exponential $\mathcal{E}(\pi \cdot S)$ of a process $\pi \cdot S$ for $\pi \in \mathcal{H}^s$ can be given:

$$\mathcal{E}(\pi \cdot S)_t \triangleq \exp\left((\pi \cdot S)_t - \frac{1}{2}\int_0^t \pi_s^2 d\,[S]_t^c\right) \prod_{s \leq t}(1 + \pi_s \Delta S_s)\exp(-\pi_s \Delta S_s), \qquad (14)$$

provided that condition (1) - the existence of a finite quadratic variation - holds. In the manner of Subsection 2.4, the process $W_t^\pi \triangleq \mathcal{E}(\pi \cdot S)_t$ can now be interpreted as the evolution of the wealth of an investor who invests the proportion π_t of her/his total wealth at time t in the risky asset.

3.2 A Counterexample

The goal of this subsection is to show that the results of Section 2. cannot be extended to the class of processes with jumps, not even in the case when the process S is obtained from a semimartingale via an enlargement of filtration. More precisely, we construct two filtrations $\mathbb{F} \subseteq \mathbb{G}$, and an $\mathbb{F}$-semimartingale S with the following properties:

$$\left.\begin{array}{ll} \text{(NS)} & S \text{ is } \textit{not} \text{ a } \mathbb{G}\text{-semimartingale, but} \\ \text{(FL)} & \sup\limits_{\pi \in \mathcal{H}^s(\mathbb{G})} \mathbb{E}\left[\log(W_1^\pi)\right] < \infty \end{array}\right\}. \qquad (15)$$

Before giving the details of our construction let us pause and try to explain the intuition behind the example. The central idea is that the introduction of jumps into the dynamics of the stock price can lead to a drastic restriction of the set of portfolios at the disposal of a logarithmic utility maximizer. Simply, any portfolio leading to a negative terminal wealth with positive probability yields an expected utility of negative infinity (as usual, we set $\log(x) = -\infty$ for $x \leq 0$), and is, therefore, clearly inferior to the constant portfolio $\pi \equiv 0$. Suppose that the process S jumps in an unpredictable fashion, while its continuous part fails the semimartingale property "just barely". In that case, we are able to envision the situation in which the non-semimartingality of

S cannot be exploited for unbounded gains in logarithmic utility due to previously mentioned scarcity of useful portfolio strategies. In other words, any strategy that might lead to a large wealth suffers from the risk of finishing negative with positive probability.

Theorem 7.2 in [5] ensures the semimartingality of the price process provided it is locally bounded and satisfies the no free lunch with vanishing risk for buy-and-hold strategies. Moreover, Example 7.5 (also in [5]) illustrates that the condition of local boundedness cannot be relaxed. The idea of this example is similar to the above; namely, the set of admissible portfolios may be almost empty.

Our construction of the process S utilizes the following ingredients:

1. B is a Brownian motion and $\mathbb{F}^B \triangleq \{\mathcal{F}^B_t\}_{t\in[0,1]}$ is the (right-continuous and complete) augmentation of the filtration generated by B.
2. M is the Gaussian martingale given by $M_t \triangleq \int_0^t \sigma(u)\, dB_u$, where
$$\sigma(t) \triangleq \frac{|\log(1-t)|^{-2/3}}{\sqrt{1-t}} \mathbf{1}_{\{1>t>\frac{1}{2}\}}.$$
3. N^1 and N^2 are two independent Poisson processes (of course also independent of the Brownian motion B).
4. N is the pure-jump process defined by $N_t \triangleq N^1_t - N^2_t$ and $\mathbb{F}^N$ is the filtration generated by the process N (or, equivalently, by N^1 and N^2).

Having introduced the necessary ingredients, the process S, announced in (15), is defined by

$$S_t \triangleq M_t + \int_0^t \frac{1}{1-u}\, dN_u,\ t \in [0,1]\,.$$

S is clearly an $\mathbb{F}$-semimartingale, where $\mathbb{F}$ is the filtration generated by B and N, i.e. $\mathbb{F} \triangleq \mathbb{F}^B \vee \mathbb{F}^N$. Let the enlarged filtration $\mathbb{G}$ be defined by adding the information about the terminal value B_1 of the Brownian motion B to $\mathbb{F}$, i.e. $\mathcal{G}_t \triangleq \mathcal{F}_t \vee \sigma(B_1)$, $t \in [0,1]$. The properties (NS) and (FL) in (15), are now established through the following lemmas.

Lemma 1 *Property (NS) in (15) holds true: S is* not *a $\mathbb{G}$-semimartingale.*

Proof It is enough to show that $\{M_t\}_{t\in[0,1]}$ is not a $\mathbb{G}$-semimartingale. This is, however, exactly the content of Theorem IV.7 in [9] and the example following it. $\Diamond$

Lemma 2 *Let $\pi \in \mathcal{H}^s(\mathbb{G})$ be a simple integrand and let W^π be the corresponding wealth process, as defined in (14). If $\mathbb{P}\,[W^\pi_1 > 0] = 1$ then*

$$\pi_t \in \big(-(1-t), 1-t\big),\ (\lambda \otimes \mathbb{P})\text{-}a.e.,$$

where λ denotes the Lebesgue measure on $[0,1]$.

Before proving Lemma 2, we require the following result:

Lemma 3 *Let N be a difference of two independent $\mathbb{G}$-Poisson processes, and let β be a $\mathbb{G}$-predictable process taking values in the set $\{-1, 1\}$. Then the process N, defined by the integral $\tilde{N}_t \triangleq \int_0^t \beta_s \, dN_s$ can be decomposed into a difference of two independent $\mathbb{G}$-Poisson processes.*

Proof Let $N_t \triangleq N_t^+ - N_t^-$ be the decomposition of N into two independent Poisson processes, and let $\beta_t^+ \triangleq \max(\beta_t, 0)$ and $\beta_t^- \triangleq \max(-\beta_t, 0)$ so that $\beta_t = \beta_t^+ - \beta_t^-$ and $\beta_t^+ + \beta_t^- = 1$, for all $t \in [0, 1]$, a.s. The processes $\tilde{N}^+$ and $\tilde{N}^-$ defined by

$$\tilde{N}_t^+ \triangleq \int_0^t \beta_s^+ \, dN_s^+ + \int_0^t \beta_s^- \, dN_s^-, \text{ and}$$
$$\tilde{N}_t^- \triangleq \int_0^t \beta_s^- \, dN_s^+ + \int_0^t \beta_s^+ \, dN_s^-.$$

have the following properties

1. $\tilde{N}^+$ and $\tilde{N}^-$ are non-decreasing processes and increase only by jumps of magnitude 1.
2. $\tilde{N}_t^+ - (\beta_t^+ + \beta_t^-)t = \tilde{N}_t^+ - t$ and $\tilde{N}_t^- - (\beta_t^+ + \beta_t^-)t = \tilde{N}_t^- - t$ are martingales.
3. The intersection of the sets of jump-times for $\tilde{N}^+$ and $\tilde{N}^-$ is empty, a.s.

Items (1) and (2) imply that $\tilde{N}^+$ and $\tilde{N}^-$ are $\mathbb{G}$-Poisson processes and (3) is enough to conclude that they are independent (see [4]). Therefore, $\tilde{N} = \tilde{N}^+ - \tilde{N}^-$ is a difference of two Poisson processes. $\Diamond$

Proof (Of Lemma 2) Let the process $\hat{\pi}$ be defined as $\hat{\pi}_t \triangleq \pi_t/(1-t)1_{\{t<1\}}$, and suppose that the predictable set $A \triangleq \{(t, \omega) \in [0, 1] \times \Omega : |\hat{\pi}_t(\omega)| \geq 1\}$ satisfies $(\lambda \otimes \mathbb{P})[A] > 0$. The expression (14) for the wealth W_1^π can be split into two factors, one of which is an exponential and the other is the product of the form

$$Y \triangleq \prod_{s \leq 1} (1 + \pi_s \frac{1}{1-s} \Delta N_s) = \prod_{s \leq 1} (1 + \hat{\pi}_s \Delta N_s).$$

The sign of W_1^π is equal to the sign of Y, so in order to reach a contradiction, it will be enough to prove that $\mathbb{P}[Y \leq 0] > 0$.

Define the process $\tilde{N}_t \triangleq \int_0^t \operatorname{sgn}(\hat{\pi}_s) \, dN_s$, where $\operatorname{sgn}(x) = 1$ for $x \geq 0$ and $\operatorname{sgn}(x) = -1$, otherwise. By Lemma 3, there exist two independent Poisson processes $\tilde{N}^+$ and $\tilde{N}^-$ such that $\tilde{N} = \tilde{N}^+ - \tilde{N}^-$, and

$$Y = \prod_{s \leq 1} (1 + |\hat{\pi}_s| \, \Delta \tilde{N}_s).$$

Let J be the event that $\tilde{N}^-$ jumps exactly once on the set A, i.e.,

$$J \triangleq \left\{ \int_0^1 \mathbf{1}_A(s) \, d\tilde{N}_s^- = 1 \right\}.$$

Since $|\hat{\pi}_s| \geq 1$ on A, it is easy to see that $\mathbb{P}[Y \leq 0] \geq \mathbb{P}[J]$. In order to show that $\mathbb{P}[J] > 0$, we first define $J' \triangleq \left\{\int_0^1 \mathbf{1}_A(s)\, d\tilde{N}_s^- \geq 1\right\} \supseteq J$. The martingale property of the process $X_t \triangleq \int_0^t \mathbf{1}_A(s)\, d\tilde{N}_s^- - \int_0^t \mathbf{1}_A(s)\, ds$ implies that

$$\mathbb{E}\left[\int_0^1 \mathbf{1}_A(s)\, d\tilde{N}_s^-\right] = \mathbb{E}\left[\int_0^1 \mathbf{1}_A(s)\, ds\right] = (\lambda \otimes \mathbb{P})\,[A] > 0,$$

showing that the $\mathbb{N} \cup \{0\}$-valued random variable $\int_0^t \mathbf{1}_A(s)\, d\tilde{N}_s^-$ has a strictly positive expectation, and thus $\mathbb{P}[J'] > 0$. Define τ^1 to be the first jump time of the process $\tilde{N}^-$. By the $\mathbb{G}$-Lévy property of the Poisson process $\tilde{N}^-$, the process $\hat{N}_t \triangleq \tilde{N}^-_{\tau_1+t} - \tilde{N}^-_{\tau_1}$ is a Poisson process, independent of $\mathcal{G}_{\tau_1}$. The probability that $\hat{N}$ will stay constant for one unit of time is strictly positive, and, consequently, so is the probability that $\tilde{N}^-$ will jump exactly once on A. This implies that $\mathbb{P}[Y \leq 0] > 0$ - a contradiction. $\Diamond$

Proposition 2 *There exists a constant $C < \infty$ such that*

$$\mathbb{E}\left[\log(W_1^\pi)\right] \leq C, \text{ for all } \pi \in \mathcal{H}^s(\mathbb{G}). \tag{16}$$

Proof By Lemma 2, it is enough to show that (16) is true for all $\pi \in \mathcal{H}^s(\mathbb{G})$, with the additional property that $|\pi_s| < (1-s)$, $\lambda \otimes \mathbb{P}$-a.e.

The expression for W_1^π given in (14) factorizes into an exponential and a product of transformed jumps, so that $\mathbb{E}[\log(W_1^\pi)] \leq C(\pi) + J(\pi)$, where

$$C(\pi) \triangleq \mathbb{E}\left[\log(\mathcal{E}(\pi \cdot M))_1\right] \quad \text{and}$$

$$D(\pi) \triangleq \mathbb{E}\left[\sum_{s \leq 1} \log(1 + \frac{\pi_s}{1-s} \varDelta N_s)\right] \leq \mathbb{E}\left[\sum_{s \leq 1} \frac{\pi_s}{1-s} \varDelta N_s\right] = 0.$$

To obtain a bound on $C(\pi)$ we first apply Jensen's inequality and then Fatou's Lemma to obtain

$$C(\pi) \leq \log(\mathbb{E}\left[\mathcal{E}(\pi \cdot M)_1\right]) \leq \liminf_{t \to 1} \log(\mathbb{E}\left[\mathcal{E}(\pi \cdot M)_t\right]).$$

Now, all we need is a uniform bound (in π and t) on $\mathbb{E}[\mathcal{E}(\pi \cdot M)_t]$, for $t < 1$. This is accomplished by noting that the process M is a $\mathbb{G}$-semimartingale on any interval $[0, u]\,, u < 1$, with the semimartingale decomposition $M = \hat{M} + (M - \hat{M})$, where the $\mathbb{G}$-martingale $\hat{M}$ is given by :

$$\hat{M}_t \triangleq \int_0^t \sigma(u) \left(dB_u - \frac{B_1 - B_u}{1-u} du\right).$$

This allows us to write

$$\begin{aligned}
\mathcal{E}(\pi \cdot M)_t &= \mathcal{E}\left(\int_0^t \pi_u\, d\hat{M}_u + \int_0^t \pi_u \sigma(u) \left(\frac{B_1 - B_u}{1-u}\right) du\right) \\
&= \exp\left((\pi \cdot \hat{M})_t - (\pi^2 \cdot \left[\hat{M}\right])_t\right) \\
&\quad \times \exp\left(\frac{1}{2} \int_0^t \pi_u^2 \sigma(u)^2 du + \int_0^t \pi_u \sigma(u) \left(\frac{B_1 - B_u}{1-u}\right) du\right).
\end{aligned}$$

The Cauchy-Schwartz inequality, combined with the observation that the square of the exponential $\exp\left((\pi \cdot \hat{M})_t - (\pi^2 \cdot \left[\hat{M}\right])_t\right)$ is a positive local martingale and, hence, a supermartingale, yields:

$$\mathbb{E}\left[\mathcal{E}(\pi \cdot M)_t\right]^2 \leq \mathbb{E}\left[\exp\left(\int_0^t \pi_u^2 \sigma(u)^2 du + 2\int_0^t \pi_u \sigma(u)\left(\frac{B_1 - B_u}{1-u}\right) du\right)\right].$$

To see that this expectation can be bounded away from ∞, independently of t and π, we can use the bound $|\pi_t| \leq 1 - t$, the explicit form of the function σ, and the fact that all exponential moments of the random variable $\sup_{t\in[0,1]} |B_t|$ are finite. $\Diamond$

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References

1. Ankirchner, S., Imkeller, P.: Finite utility on financial markets with asymmetric information and structure properties of the price dynamics, Annales de l'Institut Henri Poincaré (B) Probabilités et statistiques **41**, 479–503 (2005)
2. Björk, T., Hult, H.: A note on Wick products and the fractional Black-Scholes model, Finance and Stochastics **9**, 197–209 (2005)
3. Biagini, F., Øksendal, B.: A general stochastic calculus approach to insider trading, Applied Mathematics and Optimization **52**, 167–181 (2005)
4. Brémaud, P.: Point processes and queues. New York: Springer-Verlag 1981
5. Delbaen, F., Schachermayer, W.: A general version of the fundamental theorem of asset pricing, Mathematische Annalen **300**, 463–520 (1994)
6. Di Nunno, G., Meyer-Brandis, T., Øksendal, B., Proske, F.: Optimal portfolio for an insider in a market driven by Levy processes, Quantitative Finance **6**, 83–94 (2006)
7. Karatzas, I., Pikovsky, I.: Anticipative portfolio optimization, Advances in applied probability **28**, 1095–1122 (1996)
8. Karatzas, I., Shreve, S. E.: Brownian motion and stochastic calculus, second edition, New York: Springer-Verlag 1991
9. Protter, P. E.: Stochastic integration and differential equations, second edition, New York: Springer-Verlag 2004
10. Revuz, D., Yor, M.: Continuous Martingales and Brownian Motion, third edition, New York: Springer-Verlag 1999